MTH 4436 HW Set 2.1

Fall 2021

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Set 2.1

1.a. A number is triangular if and only if it is of the form $\frac{n(n+1)}{2}$.

This is tantamount to saying that $\forall n \in \mathbb{N}$, $1+2+3+\ldots+n = \frac{n(n+1)}{2}$.

We proved this by induction (p. 5 #1.a.)

Alternatively: We can use the "dot diagram proof" that was done in class.

1.c. The sum, of any two consecutive triangular numbers is a perfect square.

Note that $t_{n+1} = t_n + (n+1)$. (Geometrically, the $n+1^{st}$ triangle is formed by taking the n^{th} triangle and adding a row containing $n+1$ dots.)

Combine this with the result from part a) that the n^{th} triangular number, $t_n = \frac{n(n+1)}{2}$, and we have:

$$t_n + t_{n+1} = t_n + [t_n + (n+1)] = 2t_n + n + 1 = 2\frac{n(n+1)}{2} + n + 1 =$$

$$n(n+1) + n + 1 = (n+1)(n+1) = (n+1)^2$$

i.e., $t_n + t_{n+1} = (n+1)^2$

Hence, the sum of any two consecutive triangular numbers is a perfect square.

Alternatively: We can use the "dot diagram proof" that was done in class.

7. Show that the difference between the squares of two consecutive triangular numbers is always a cube.

Proof:

Recall: $t_n = \frac{n(n+1)}{2}$ is the n$^{\text{th}}$ triangular number.

Consequently: $t_{n+1} = \frac{(n+1)(n+2)}{2}$

Consider: $t_{n+1}^2 - t_n^2 = \left(\frac{(n+1)(n+2)}{2}\right)^2 - \left(\frac{n(n+1)}{2}\right)^2$

$$= \underbrace{\left(\frac{(n+1)(n+2)}{2} + \frac{n(n+1)}{2}\right)\left(\frac{(n+1)(n+2)}{2} - \frac{n(n+1)}{2}\right)}_{\text{Difference of Perfect Squares}}$$

$= \left(\frac{(n+1)[(n+2)+n]}{2}\right)\left(\frac{(n+1)[(n+2)-n]}{2}\right) = \left(\frac{(n+1)(2n+2)}{2}\right)\left(\frac{(n+1)(2)}{2}\right)$

$= \left(\frac{(n+1)(n+1)(2)}{2}\right)\left(\frac{(n+1)(2)}{2}\right) = (n+1)(n+1)(n+1) = (n+1)^3$

i.e., $t_{n+1}^2 - t_n^2 = s_{n+1}^3$