MTH 4436 Test #1 - Solutions

Fall 2021

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1. Define the following: a **divides** b (i.e., $a|b$)

   Let a and b be integers, with $a \neq 0$. Then b is said to be *divisible* by a (or a *divides* b), denoted $a|b$, if there exists an integer c such that $b = ac$. If b is NOT divisible by a, (or if a does NOT divide b) then we write $a \nmid b$.

2. Define the following: **Greatest Common Divisor** of a and b (i.e., $\gcd(a, b)$)

   Let a and b be integers, with at least one of them not equal to zero. The *greatest common divisor* of a and b, denoted $\gcd(a, b)$, is the positive integer d satisfying the following:

   (a) $d|a$ and $d|b$

   (b) If $c|a$ and $c|b$, then $c \leq d$.

3. Define the following: **Relatively Prime**

   Let a and b be integers, with at least one of them not equal to zero. Then a and b are said to be *relatively prime* exactly when $\gcd(a, b) = 1$.

   Alternatively, a and b are said to be *relatively prime* exactly when a and b have no common divisors larger than 1.

4. State the Well Ordering Principle

   Every non-empty set of non-negative integers (natural numbers) has a least (smallest) element.

   Alternativley, let S be a non-empty set of non-negative integers (natural numbers). Then $\exists\ x \in S$ such that $\forall s \in S$, $x \leq s$.

5. State Pascal's Rule

   $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$; for $1 \leq k \leq n$

6. State the Binomial Theorem

   For any real numbers a and b, and any natural number n, the following holds:

   $$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i$$

   Or equivalently:

   For any real numbers a and b, and any natural number n, the following holds:

   $$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{i} b^{n-i}$$

7. State the Division Algorithm

   Given integers a and b, with $b > 0$, there exist unique integers q and r such that

   $$a = qb + r \quad \text{with } 0 \leq r < b.$$

   q is called the *quotient* and r is called the *remainder.*

8. Prove: If $a|b$ and $c|d$, then $ac|bd$.

   Proof.

   Suppose that $a|b$ and $c|d$. The there exist integers, m and n, such that $b = am$ and $d = cn$.

   Thus we have $bd = (am)(cn) = (ac)(mn)$

   i.e., $bd = (ac)(mn)$

   Hence, $ac|bd$. ■

9. Prove by Induction:

$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$ (i.e., $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$)

Proof.

Show true for $n = 1$

$\sum_{i=1}^{1} i = 1 = \frac{(1)[(1)+1]}{2}$

i.e., $\sum_{i=1}^{1} i = \frac{(1)[(1)+1]}{2}$

Assume true for $n = k$; show true for $n = k + 1$

i.e., Assume that $\underbrace{\sum_{i=1}^{k} i = \frac{k(k+1)}{2}}_{\text{(induction hypothesis)}}$ is true, and show that this implies that

$\sum_{i=1}^{k+1} i = \frac{(k+1)[(k+1)+1]}{2}$ is true. (i.e., show that $\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$ is true.)

Observe: $\sum_{i=1}^{k+1} i = \left(\sum_{i=1}^{k} i\right) + (k+1) = \frac{k(k+1)}{2} + (k+1) = \left(\frac{k}{2} + 1\right)(k+1)$

by induction hypothesis

$$= \left(\frac{k}{2} + \frac{2}{2}\right)(k+1) = \left(\frac{k+2}{2}\right)(k+1) = \frac{(k+2)(k+1)}{2} = \frac{(k+1)(k+2)}{2}.$$

i.e., $\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$.

Thus, $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ for all natural numbers, n. ■

10. Prove:

$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \ldots + (-1)^n \binom{n}{n} = 0$

Proof. By the Binomial Theorem, $(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i$

If we let $a = 1$, and $b = -1$, then we have

$0 = (1-1)^n = \sum_{i=0}^{n} \binom{n}{i} 1^{n-i} (-1)^i$

$$\begin{aligned} \Rightarrow \quad 0 \quad = \quad \sum_{i=0}^{n} \binom{n}{i} \quad &= \quad (-1)^0 \binom{n}{0} + (-1)^1 \binom{n}{1} + (-1)^2 \binom{n}{2} + \ldots + (-1)^n \binom{n}{n} \\ &= \quad \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \ldots + (-1)^n \binom{n}{n} \end{aligned}$$

i.e., $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \ldots + (-1)^n \binom{n}{n} = 0$ ■

11. Prove: The sum of any two consecutive triangular numbers is a perfect square.

Proof. Recall that the n^{th} triangular number t_n is given by:

$t_n = \frac{n(n+1)}{2}$

Thus, for any natural number n, the sum of two consecutive triangular numbers is given by:

$t_n + t_{n+1} = \frac{n(n+1)}{2} + \frac{(n+1)(n+2)}{2} = \frac{(n+1)[n+(n+2)]}{2} = \frac{(n+1)(2n+2)}{2} = (n+1)^2$

i.e., $t_n + t_{n+1} = (n+1)^2$ ■

Alternatively: the result can be shown using "dot diagrams."

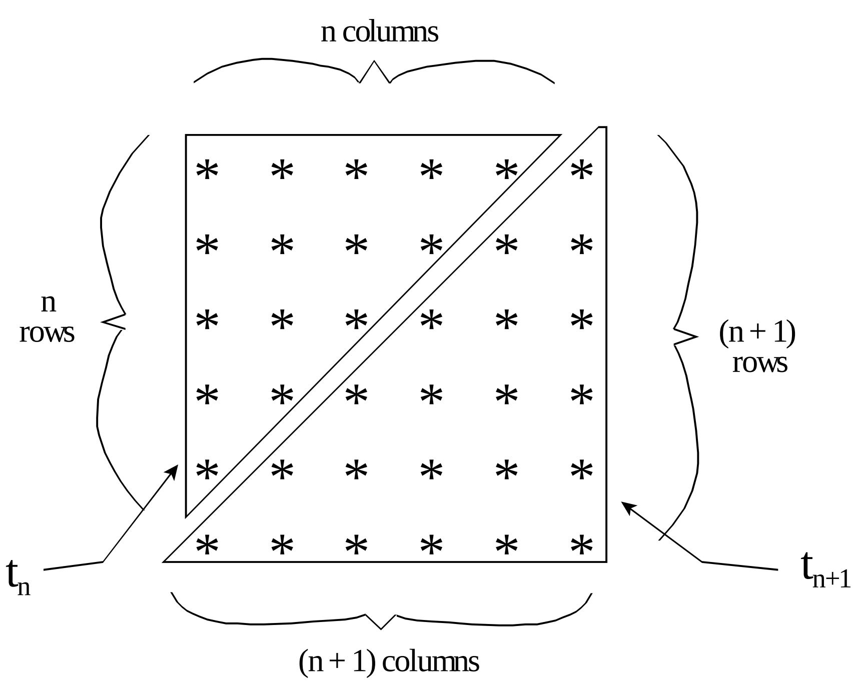
With reference to the diagram below:

t_n is the number of dots in the right isosceles triangle having legs of length n.

t_{n+1} is the number of dots in the right isosceles triangle having legs of length $(n+1)$.

When we add t_n and t_{n+1} it's the same as putting the two triangles together to form a square having $(n+1)$ rows with $(n+1)$ dots in each row.

i.e. $t_n + t_{n+1} = (n+1)^2$



12. Prove: If $a|b$ and $a|c$, then $a|(bx+cy)$ for arbitrary integers, x and y.

Proof. Suppose that $a|b$ and $a|c$.

Then there exist integers, m and n, such that $b = am$ and $c = an$.

Thus, $bx + cy = (am)x + (an)y = a(mx + ny)$

i.e., $bx + cy = a(mx + ny)$

Thus, $a|(bx + cy)$ ■

13. Prove that the cube of an integer n is either of the form $9k, 9k+1$, or $9k+8$.

Proof. By the Division Algorithm, with reference to 3 as the divisor, we need only consider 3 cases:

$\boxed{n = 3m \text{ for some } m \in \mathbf{Z}}$

In this case, $n^3 = (3m)^3 = 27m^3 = 9\underbrace{(3m^3)}_{k} = 9k.$

i.e., $n^3 = 9k \quad$ for $k = 3m^3$

$\boxed{n = 3m + 1 \text{ for some } m \in \mathbf{Z}}$

In this case, $n^3 = (3m+1)^3 = 27m^3+27m^2+9m+1 = 9\underbrace{(3m^3+3m^2+m)}_{k}+1 = 9k+1.$

i.e., $n^3 = 9k + 1 \quad$ for $k = 3m^3 + 3m^2 + m$

$\boxed{n = 3m + 2 \text{ for some } m \in \mathbf{Z}}$

In this case, $n^3 = (3m+2)^3 = 27m^3 + 54m^2 + 36m + 8 = 9\underbrace{(3m^3 + 6m^2 + 4m)}_{k} + 8$

$= 9k + 8.$

i.e., $n^3 = 9k + 8 \quad$ for $k = 3m^3 + 6m^2 + 4m$

After having exhausted all possible cases, we find only three possibilities:

$n^3 = 9k$, $n^3 = 9k + 1$, or $n^3 = 9k + 8$. ■

Extra! (10 pts) Prove that the sum of the squares of two odd integers cannot be a perfect square.

Two arbitrary odd integers can be represented as $2k+1$ and $2j+1$.

The sum of their squares is $(2k+1)^2 + (2j+1)^2 = 4k^2 + 4k + 1 + 4j^2 + 4j + 1 =$

$4k^2 + 4k + 4j^2 + 4j + 2 = 4\left(k^2 + k + j^2 + j\right) + 2 = 4m + 2,$ where $m = k^2 + k + j^2 + j$

The point here, is that the sum of the squares of any two odd numbers is of the form $4m+2$.

Could this be a perfect square? Well, let's see what form(s) perfect squares can have.

An even number $2n$, when squared, has the form: $4n^2 = 4m,$ where $m = n^2$.

An odd number $2n+1$, when squared, has the form: $4n^2+4n+1 = 4m+1,$ where $m = n^2+n$.

What we have learned is that perfect squares are either of the form $4m$ or $4m+1$. Perfect squares are *never* of the form $4m+2$, which is the form that the sum of the squares of two odd numbers always has.

Hence, the sum of the squares of two odd integers cannot be a perfect square. ■

Extra! (10 pts) Prove that for any integer n, one of the integers n, $n+2$, $n+4$ is divisible by 3.

14. **Proof.** Let $n \in \mathbb{N}$ be given.

    By the Division Algorithm, n must have exactly one of the following three forms:

    $n = 3k$ for some $k \in \mathbf{Z}$
    $n = 3k+1$ for some $k \in \mathbf{Z}$
    $n = 3k+2$ for some $k \in \mathbf{Z}$

    Case #1 $n = 3k$.

    In this case, our claim is established, since $3|n$.

    Case #2 $n = 3k+1$.

    In this case, $n+2 = (3k+1)+2 = 3k+3 = 3(k+1)$. i.e., $n+2$ is divisible by 3.

    Case #3 $a = 3k+2$.

    In this case, $n+4 = (3k+2)+4 = 3k+6 = 3(k+2)$. i.e., $n+4$ is divisible by 3. ■