

Integrals and Natural Logarithms #7 - Solutions

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Name _____

1. Compute: $\int (2x^5 - 6x^3 + 4x + 8) dx =$

$$\begin{aligned}\int (2x^5 - 6x^3 + 4x + 8) dx &= 2 \left[\frac{x^6}{6} \right] - 6 \left[\frac{x^4}{4} \right] + 4 \left[\frac{x^2}{2} \right] + 8x + C \\ &= \frac{1}{3}x^6 - \frac{3}{2}x^4 + 2x^2 + 8x + C\end{aligned}$$

i.e., $\int (2x^5 - 6x^3 + 4x + 8) dx = \frac{1}{3}x^6 - \frac{3}{2}x^4 + 2x^2 + 8x + C$
Don't forget the "+C"

2. Compute: $\int (5 \cos(x) - 7 \sec^2(x)) dx =$

$$\int (5 \cos(x) - 7 \sec^2(x)) dx = 5 [\sin(x)] - 7 [\tan(x)] + C$$

i.e., $\int (5 \cos(x) - 7 \sec^2(x)) dx = 5 \sin(x) - 7 \tan(x) + C$
Don't forget the "+C"

3. Compute: $\int_{x=0}^{x=2} (8x^3 + 9x^2 + 2x) dx =$

$$\begin{aligned}\int_{x=0}^{x=2} \underbrace{(8x^3 + 9x^2 + 2x)}_{f(x)} dx &= \underbrace{\left[8 \left(\frac{x^4}{4} \right) + 9 \left(\frac{x^3}{3} \right) + 2x \right]_{x=0}^{x=2}}_{F(x)} = \underbrace{\left[2x^4 + 3x^3 + 2x \right]_{x=0}^{x=2}}_{F(x)} \\ &= \underbrace{\left[2(2)^4 + 3(2)^3 + 2(2) \right]}_{F(2)} - \underbrace{\left[2(0)^4 + 3(0)^3 + 2(0) \right]}_{F(0)} = 60\end{aligned}$$

i.e., $\int_{x=0}^{x=2} (8x^3 + 9x^2 + 2x) dx = 60$

4. Compute: $\int (x^3 + x^2)^4 (21x^2 + 14x) dx$

1. Is u-sub appropriate?

a. Is there a composite function?

Yes! $(x^3 + x^2)^4$ (A function raised to a power is always a composite function!)

Let u = the “inner” of the composite function

$$\Rightarrow u = (x^3 + x^2)$$

b. Is there an (approximate) function/derivative pair?

$$\text{Yes!} \quad \underbrace{(x^3 + x^2)}_{\text{function}} - - - \rightarrow \underbrace{(21x^2 + 14x)}_{\text{deriv}}$$

Let u = the “function” of the function/deriv pair

$$\Rightarrow u = (x^3 + x^2)$$

c. Is the “function” of the function/deriv pair the same as the “inner” of the composite function?

(i.e., do criteria **a** and **b** suggest the same choice of u ?)

Yes! \Rightarrow u-substitution is appropriate

2. Compute du

$$\begin{aligned} u &= x^3 + x^2 \\ \Rightarrow \frac{du}{dx} &= 3x^2 + 2x \\ \Rightarrow du &= (3x^2 + 2x) dx \\ \Rightarrow 7du &= (21x^2 + 14x) dx \end{aligned}$$

3. Analyze in terms of u and du

$$\int \underbrace{(x^3 + x^2)^4}_{u^4} \underbrace{(21x^2 + 14x) dx}_{7du} = \int u^4 \cdot 7du = 7 \int u^4 du$$

4. Integrate (in terms of u).

$$7 \int u^4 du = 7 \left[\frac{u^5}{5} \right] + C = \frac{7}{5} u^5 + C$$

5. Re-express in terms of the original variable, x .

$$\int (x^3 + x^2)^4 (21x^2 + 14x) dx = \frac{7}{5} \underbrace{(x^3 + x^2)^5}_{\frac{7}{5} u^5 + C} + C$$

$$\text{i.e., } \int (x^3 + x^2)^4 (21x^2 + 14x) dx = \frac{7}{5} (x^3 + x^2)^5 + C$$

5. Compute: $\int \sin(\sec(x)) \sec(x) \tan(x) dx =$

1. Is u-sub appropriate?

a. Is there a composite function?

Yes! $\sin(\sec(x))$

outer inner

Let $u =$ the “inner” of the composite function

$\Rightarrow u = \sec(x)$

b. Is there an (approximate) function/derivative pair?

Yes! $\underbrace{\sec(x)}_{\text{function}} \text{ --- } \rightarrow \underbrace{\sec(x) \tan(x)}_{\text{deriv}}$

Let $u =$ the “function” of the function/deriv pair

$\Rightarrow u = \sec(x)$

c. Is the “function” of the function/deriv pair the same as the “inner” of the composite function?

(i.e., do criteria **a** and **b** suggest the same choice of u ?)

Yes! \Rightarrow u-substitution is appropriate

2. Compute du

$$\begin{aligned} u &= \sec(x) \\ \Rightarrow \frac{du}{dx} &= \sec(x) \tan(x) \\ \Rightarrow du &= \sec(x) \tan(x) dx \end{aligned}$$

3. Analyze in terms of u and du

$$\int \underbrace{\sin(\sec(x))}_{\sin(u)} \underbrace{\sec(x) \tan(x) dx}_{du} = \int \sin(u) du$$

4. Integrate (in terms of u).

$$\int \sin(u) du = -\cos(u) + C$$

5. Re-express in terms of the original variable, x .

$$\int \sin(\sec(x)) \sec(x) \tan(x) dx = \underbrace{-\cos(\sec(x)) + C}_{-\cos(u)+C}$$

$$\text{i.e., } \int \sin(\sec(x)) \sec(x) \tan(x) dx = -\cos(\sec(x)) + C$$

6. Compute: $\int \frac{6x^2+3x+3}{4x^3+3x^2+6x} dx =$

$$\int \frac{6x^2+3x+3}{4x^3+3x^2+6x} dx \underbrace{=}_{\text{re-write}} \int \frac{1}{4x^3+3x^2+6x} (6x^2 + 3x + 3) dx$$

Remark: Note that we have an approximate function/derivative pair, with the “function” in the denominator. This usually indicates that u-substitution is appropriate, with the result being a natural logarithm.

1. Is u-sub appropriate?

a. Is there a composite function?

Yes! $\frac{1}{4x^3+3x^2+6x}$ is the same as $(4x^3 + 3x^2 + 6x)^{-1}$, so it is a function raised to a power.

Let u = the “inner” of the composite function

$$\Rightarrow u = 4x^3 + 3x^2 + 6x$$

b. Is there an (approximate) function/derivative pair?

$$\text{Yes!} \quad \underbrace{(4x^3 + 3x^2 + 6x)}_{\text{function}} - - - - \rightarrow \underbrace{(6x^2 + 3x + 3)}_{\text{deriv}}$$

Let u = the “function” of the function/deriv pair

$$\Rightarrow u = 4x^3 + 3x^2 + 6x$$

c. Is the “function” of the function/deriv pair the same as the “inner” of the composite function?

(i.e., do criteria **a** and **b** suggest the same choice of u ?)

Yes! \Rightarrow u-substitution is appropriate

2. Compute du

u	$=$	$4x^3 + 3x^2 + 6x$
$\Rightarrow \frac{du}{dx}$	$=$	$12x^2 + 6x + 6$
$\Rightarrow du$	$=$	$(12x^2 + 6x + 6) dx$
$\Rightarrow \frac{1}{2} du$	$=$	$(6x^2 + 3x + 3) dx$

3. Analyze in terms of u and du

$$\int \underbrace{\frac{1}{4x^3 + 3x^2 + 6x}}_{\frac{1}{u}} \underbrace{(6x^2 + 3x + 3) dx}_{\frac{1}{2} du} = \int \frac{1}{u} \frac{1}{2} du = \frac{1}{2} \int \frac{1}{u} du$$

4. Integrate (in terms of u).

$$\frac{1}{2} \int \frac{1}{u} du = \frac{1}{2} \ln |u| + C$$

5. Re-express in terms of the original variable, x .

$$\int \frac{6x^2+3x+3}{(4x^3+3x^2+6x)} dx = \underbrace{\frac{1}{2} \ln |4x^3 + 3x^2 + 6x| + C}_{\frac{1}{2} \ln |u| + C}$$

$$\text{i.e., } \int \frac{6x^2+3x+3}{(4x^3+3x^2+6x)} dx = \frac{1}{2} \ln |4x^3 + 3x^2 + 6x| + C$$

7. Compute: $\frac{d}{dx} [\ln (\sin (x) + \cos (x))] =$

$$\underbrace{\frac{d}{dx} [\ln (\sin (x) + \cos (x))]}_{\frac{d}{dx} [\ln (g(x))]} = \underbrace{\frac{1}{\sin (x) + \cos (x)}}_{\frac{1}{g(x)}} \cdot \underbrace{(\cos (x) - \sin (x))}_{g'(x)} = \frac{\cos (x) - \sin (x)}{\sin (x) + \cos (x)}$$

i.e., $\frac{d}{dx} [\ln (\sin (x) + \cos (x))] = \frac{\cos (x) - \sin (x)}{\sin (x) + \cos (x)}$

8. Compute: $\frac{d}{dx} [\ln (8x^2 - 7x + 5)] =$

$$\underbrace{\frac{d}{dx} [\ln (8x^2 - 7x + 5)]}_{\frac{d}{dx} [\ln (g(x))]} = \underbrace{\frac{1}{8x^2 - 7x + 5}}_{\frac{1}{g(x)}} \cdot \underbrace{(16x - 7)}_{g'(x)} = \frac{16x - 7}{8x^2 - 7x + 5}$$

i.e., $\frac{d}{dx} [\ln (8x^2 - 7x + 5)] = \frac{16x - 7}{8x^2 - 7x + 5}$

9. Compute: $\frac{d}{dx} \left[\ln \left(\sqrt{\sin (x) \cos (x)} \right) \right] = \frac{d}{dx} \left[\ln \left((\sin (x) \cos (x))^{\frac{1}{2}} \right) \right]$

Remark: We can compute this derivative directly, in it's current form, but it would be much easier to use the *algebraic properties* of natural logarithms to simplify the expression first.

$$\frac{d}{dx} \left[\ln \left((\sin (x) \cos (x))^{\frac{1}{2}} \right) \right] = \underbrace{\frac{d}{dx} \left[\frac{1}{2} \ln (\sin (x) \cos (x)) \right]}_{\ln (a^n) = n \ln (a)} = \underbrace{\frac{d}{dx} \left[\frac{1}{2} (\ln (\sin (x)) + \ln (\cos (x))) \right]}_{\ln (ab) = \ln (a) + \ln (b)}$$

NOW we're ready to compute the derivative!

$$\begin{aligned} \frac{d}{dx} \left[\ln \left((\sin (x) \cos (x))^{\frac{1}{2}} \right) \right] &= \frac{d}{dx} \left[\frac{1}{2} [\ln (\sin (x)) + \ln (\cos (x))] \right] = \frac{1}{2} \frac{d}{dx} [\ln (\sin (x)) + \ln (\cos (x))] \\ &= \frac{1}{2} \left[\frac{1}{\sin (x)} \cdot \cos (x) + \frac{1}{\cos (x)} (-\sin (x)) \right] = \frac{1}{2} (\cot (x) - \tan (x)) \end{aligned}$$

i.e., $\frac{d}{dx} \left[\ln \left(\sqrt{\sin (x) \cos (x)} \right) \right] = \frac{1}{2} (\cot (x) - \tan (x))$

10. Compute: $\int_{x=0}^{x=3} \sqrt{x+1} dx = \int_{x=0}^{x=3} (x+1)^{\frac{1}{2}} dx$

1. Is u-sub appropriate?

a. Is there a composite function?

Yes! $(x+1)^{\frac{1}{2}}$ (A function raised to a power is *always* a composite function!)

Let u = the “inner” of the composite function

$$\Rightarrow u = (x+1)$$

b. Is there an (approximate) function/derivative pair?

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c. Is the “function” of the function/deriv pair the same as the “inner” of the composite function?

(i.e., do criteria **a** and **b** suggest the same choice of u ?)

No – we don’t see an (approximate) function/derivative pair.

Nevertheless, we will proceed, based on the recommendation of Criterion a. (Note: Since Criterion b was not satisfied, u-substitution may not work.)

2. Compute du

$$\begin{array}{rcl} u & = & x+1 \\ \Rightarrow \frac{du}{dx} & = & 1 \\ \Rightarrow du & = & 1 dx \\ \text{i.e., } du & = & dx \end{array}$$

When $x = 0$, $u = x+1 = (0)+1 = 1$

When $x = 3$, $u = x+1 = (3)+1 = 4$

3. Analyze in terms of u and du

$$\int_{x=0}^{x=3} \underbrace{(x+1)^{\frac{1}{2}}}_{u^{\frac{1}{2}}} \underbrace{dx}_{du} = \int_{u=1}^{u=4} u^{\frac{1}{2}} du$$

Don’t forget to re-write the limits of integration in terms of u !

4. Integrate (in terms of u).

$$\int_{u=1}^{u=4} u^{\frac{1}{2}} du = \left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}} \right]_{u=1}^{u=4} = \left[\frac{2}{3} u^{\frac{3}{2}} \right]_{u=1}^{u=4} = \underbrace{\frac{2}{3} (4)^{\frac{3}{2}}}_{F(4)} - \underbrace{\frac{2}{3} (1)^{\frac{3}{2}}}_{F(1)} = \frac{2}{3} (8) - \frac{2}{3} (1) = \frac{14}{3}$$

$$\text{i.e., } \int_{x=0}^{x=3} \sqrt{x+1} dx = \frac{14}{3}$$

Remark: It turns out, that we really DID have an (approximate) function/derivative pair.

$$\int_{x=0}^{x=3} (x+1)^{\frac{1}{2}} dx = \text{ is the same as } \int_{x=0}^{x=3} (x+1)^{\frac{1}{2}} \cdot 1 dx$$

Our (approximate) function/derivative pair is: $\underbrace{(x+1)}_{\text{function}} - - - - \rightarrow \underbrace{1}_{\text{deriv}}$