MTH 1112 Test #2 - Solutions

SUMMER 2021

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Show CLEARLY how you arrive at your answers!

1. Given that $f(x) = \frac{x}{x^2+1}$, compute the following:

   (a) $f(-1) = \frac{(-1)}{(-1)^2+1} = -\frac{1}{2}$

   $\boxed{f(-1) = -\frac{1}{2}}$

   (b) $f(2x) = \frac{(2x)}{(2x)^2+1} = \frac{2x}{4x^2+1}$

   $\boxed{f(2x) = \frac{2x}{4x^2+1}}$

   (c) $f(x+h) = \frac{(x+h)}{(x+h)^2+1} = \frac{(x+h)}{(x^2+2xh+h^2)+1} = \frac{(x+h)}{x^2+2xh+h^2+1}$

   $\boxed{f(x+h) = \frac{(x+h)}{x^2+2xh+h^2+1}}$

2. Given that $f(x) = \frac{x}{x^2-16}$, find the domain.

With a rational function, we begin by assuming that the domain is the entire set of real numbers.

Then, we exclude those real numbers that yield division by zero when we plug them into $f(x)$.

To find these numbers, we set the denominator equal to zero and solve for x.

Division by zero when: $x^2 - 16 = 0$

Division by zero when: $(x+4)(x-4) = 0$

Division by zero when: $x = -4$ or $x = 4$

The domain is: $(-\infty, 4) \cup (-4, 4) \cup (4, \infty)$

Alternatively, the domain is: $\{x|\ x \neq -4, 4\}$

The domain is $(-\infty, 4) \cup (-4, 4) \cup (4, \infty) = \{x|\ x \neq -4, 4\}$

3. Given that $f(x) = \begin{cases} -2x+3 & \text{if } x < 1 \\ 3x-2 & \text{if } x \geq 1 \end{cases}$, draw the graph of $y = f(x)$

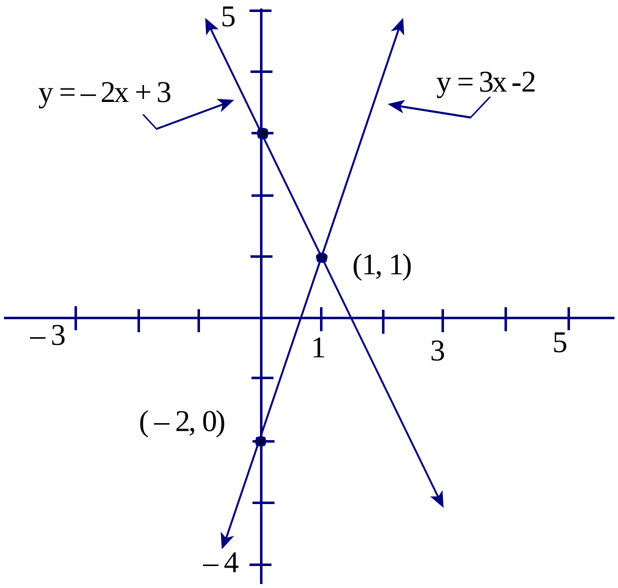
Note that the definition of $f(x)$ changes at $x = 1$.

To the left of $x = 1$, $f(x) = -2x + 3$

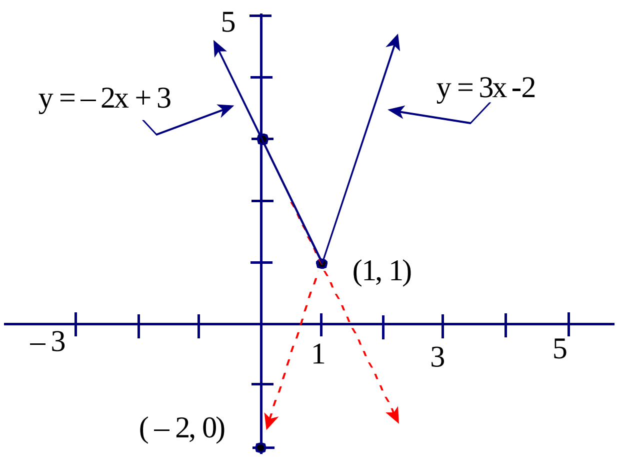
To the right of $x = 1$, $f(x) = 3x - 2$.

At $x = 1$, $f(x) = 3x - 2$

If we graph both equations ($f(x) = -2x + 3$ and $f(x) = 3x - 2$), this is what we get:



If we remove the portion of the graph of $y = 3x - 2$ that lies to the left of $x = 1$ and we remove the portion of the graph of $y = -2x + 3$ that lies to the right of $x = 1$, we have our graph of $f(x)$.



4. Given that $f(x) = \begin{cases} -2x+3 & \text{if } x < 1 \\ 3x-2 & \text{if } x \geq 1 \end{cases}$,

(a) Compute: $f(0) =$

Note that $0 < 1$

Thus, to compute $f(0)$, we use the definition of $f(x)$ that corresponds to $x < 1$

i.e., $f(x) = -2x + 3$

Therefore: $f(0) = -2(0) + 3 = 3$

$f(0) = 3$

(b) Compute: $f(2) =$

Note that $2 > 1$

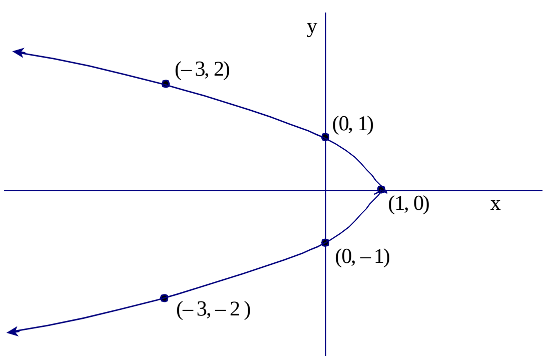
Thus, to compute $f(2)$, we use the definition of $f(x)$ that corresponds to $x \geq 1$

i.e., $f(x) = 3x - 2$

Therefore: $f(2) = 3(2) - 2 = 4$

$f(2) = 4$

5. The graph below defines a relation between variables x and y.



(a) Determine whether or not y is a function of x. Justify your answer.

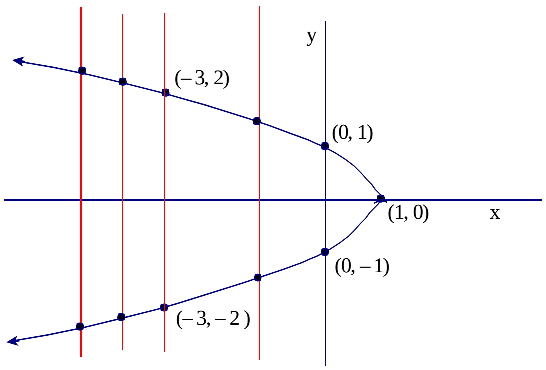
Note that for the x-value $x = -3$, there are two values of y ($y = -2$ and $y = 2$) assigned to this one value of x.

Since two distinct values of y are assigned to the same value of x, **the graph does NOT define y as a function of x.**

Alternatively: If the graph defines y as a function of x, then the graph will "pass the vertical line test" (i.e. no vertical line will intersect the graph at more than one point.)

Looking at the graph that relates y and x, we see that it possible to draw vertical lines that DO intersect the graph at more than one point.

Therefore, **the graph does NOT define y as a function of x.**



(b) State the domain of the relation.

The domain of the relation is the set of all x-values that correspond to a point in the graph.

There is at least one point on the graph for each x-value less than or equal to 1. There are NO points on the graph corresponding to x-values greater than 1.

Domain is $(-\infty, 1] = \{x|\ x \leq 1\}$

(c) State the range of the relation.

The range of the relation is the set of all y-values that correspond to a point in the graph.

There is exactly one point on the graph for each y-value from $-\infty$ to ∞.

Range is $(-\infty, \infty) = \{\text{All Real Numbers}\}$

(d) Give the x-intercepts.

The graph touches the x-axis at one point only: $(1, 0)$

x-intercept is $x = 1$

(e) Give the y-intercepts.

The graph touches the y-axis at two points: $(0, -1)$ and $(0, 1)$

y-intercepts are $y = -1$ and $y = 1$

(f) State any symmetry with respect to the x-axis, y-axis, or the origin.

The portion of the x-axis lying below the x-axis is the mirror image of the portion of the graph lying above the x-axis. Hence, symmetric with respect to the x-axis.

Alternatively: It appears to be the case that each value of y is paired with the same x-value as is $-y$. For example: $y = -2$ and $y = 2$ are both paired with $x = -3$. Also, $y = -1$ and $y = 1$ both paired with $x = -3$. Hence, we would conclude that the graph is symmetric with respect to the x-axis.

The graph is symmetric with respect to the x-axis.

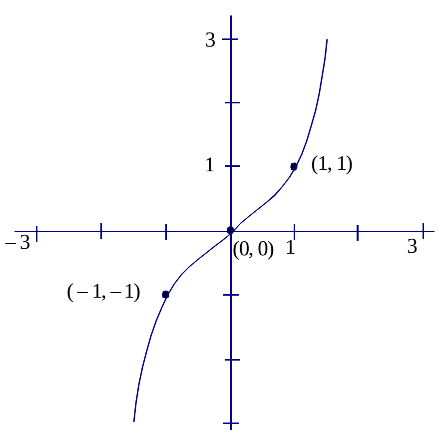
6. $f(x) = (x-1)^3 + 2$.

Graph the function using techniques of shifting, compressing/stretching, and/or reflecting. Start with the graph of the basic function and show all of the steps. Be sure to show at least three (3) key points.

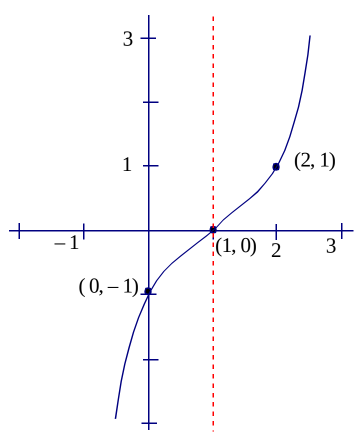
We will start by graphing the elementary function that is most closely related to $f(x) = (x-1)^3 + 2$, namely: $f(x) = x^3$

Then we'll graph $f(x) = (x-1)^3$, which is the graph of $f(x) = x^3$ shifted one unit right.

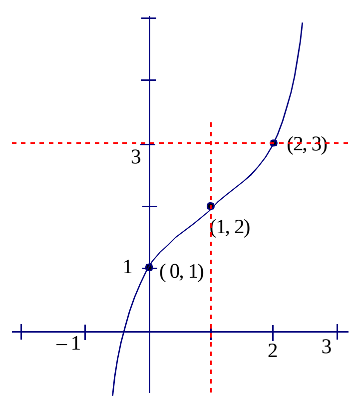
Finally, we'll graph $f(x) = (x-1)^3 + 2$, which is the graph of $f(x) = (x-1)^3$ shifted two units up.



$f(x) = x^3$



$f(x) = (x-1)^3$



$f(x) = (x-1)^3 + 2$

7. $f(x) = x^2 - 8x + 1$. Complete the square of the expression. Then graph the function.

1. Group the non constant terms together

$\Rightarrow f(x) = (x^2 - 8x) + 1$

2. Factor out the coefficient of x^2

$\Rightarrow f(x) = (x^2 - 8x) + 1$

3. Take one half the coefficient of x, square it, and add it inside the parentheses. Compensate by subtracting it outside the parentheses.

$\Rightarrow f(x) = \left(x^2 - 8x + \left(\frac{-8}{2}\right)^2\right) + 1 - \left(\frac{-8}{2}\right)^2$

i.e., $f(x) = \left(x^2 - 8x + (-4)^2\right) + 1 - 16$

4. Write the expression inside the parentheses as a perfect square.

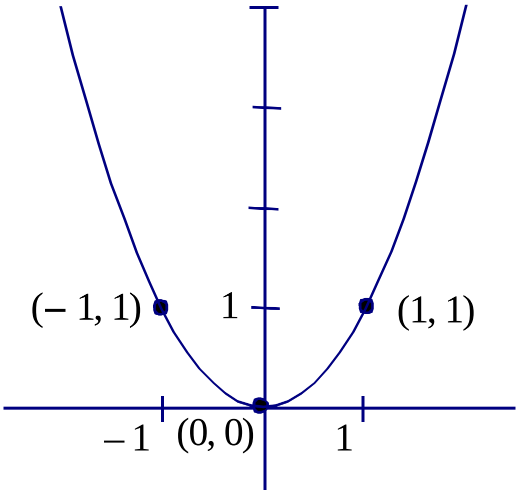
$\Rightarrow f(x) = (x-4)^2 - 15$

To graph the function $f(x) = (x-4)^2 - 15$, first we graph the closely related to the elementary function $f(x) = x^2$

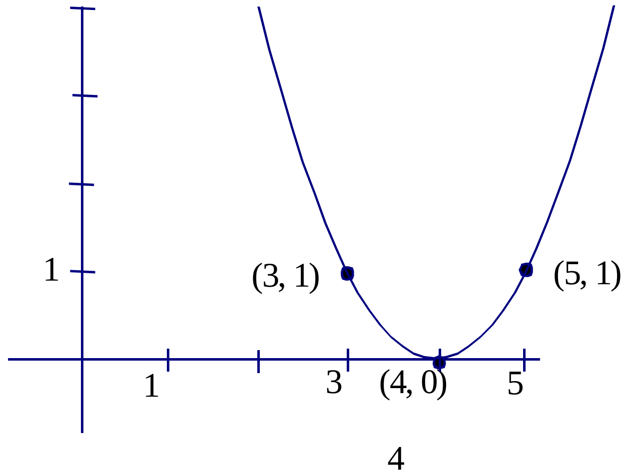
Then we graph $f(x) = (x-4)^2$, which is the graph of $f(x) = x^2$ shifted 4 units to the right.

Then we graph $f(x) = (x-4)^2 - 15$, which is the graph of $f(x) = (x-4)^2$ shifted 15 units down.

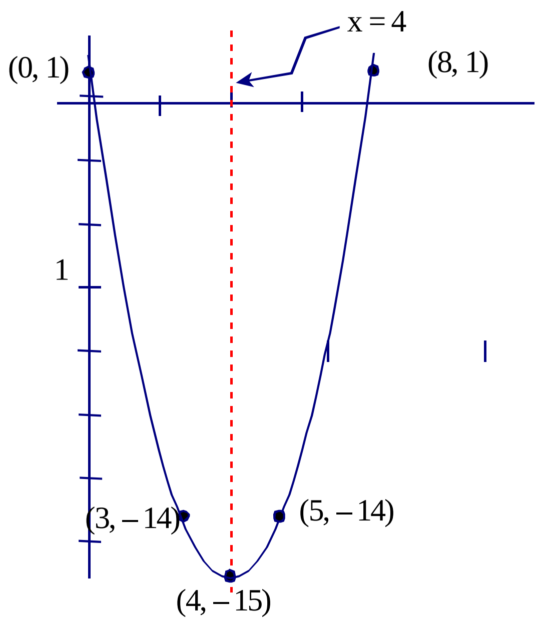
(The graphs appear on the next page)



$f(x) = x^2$

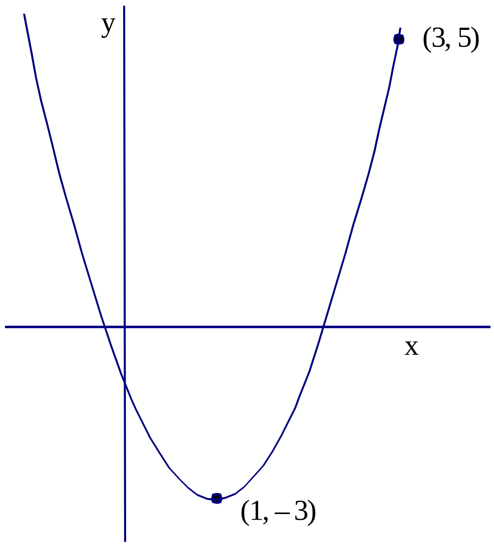


$f(x) = (x-4)^2$



$f(x) = (x-4)^2 - 15$

8. The graph of a quadratic function is given below. Based on the information contained in the graph, give the definition of the function. $f(x) =$



Recall: Given the equation of a parabola in the form: $f(x) = a(x-h)^2 + k$, the vertex is (h, k).

Since the vertex of the parabola is $(h, k) = (1, -3)$, the equation of a parabola is $f(x) = a(x-1)^2 + (-3)$

i.e., $f(x) = a(x-1)^2 - 3$

To find the value of a, we take the coordinates of any other point on the graph, other than the vertex, and plug them into the definition of $f(x)$.

We have the point $(3, 5)$.

Plugging these coordinates into $f(x) = a(x-1)^2 - 3$, yields:

$f(3) = a((3) - 1)^2 - 3 = 5$

$\Rightarrow a(2)^2 - 3 = 5$

$\Rightarrow 4a - 3 = 5$

$\Rightarrow 4a = 8$

$\Rightarrow a = 2$

Hence, $f(x) = 2(x-1)^2 - 3 = 2x^2 - 4x - 1$

$f(x) = 2(x-1)^2 - 3$ or $f(x) = 2x^2 - 4x - 1$

9. Solve the inequality: $2x^2 < 5x + 3$

Rewrite as: $2x^2 - 5x - 3 < 0$

Considering the related equation, $2x^2 - 5x - 3 = 0$, we notice that it fits the form:

$ax^2 + bx + c = 0$, with $a = 2$

Since $a > 0$, the parabola "opens upward."

Also: if we factor the left hand side of the equation $2x^2 - 5x - 3 = 0$, we have:

$(2x + 1)(x - 3) = 0$

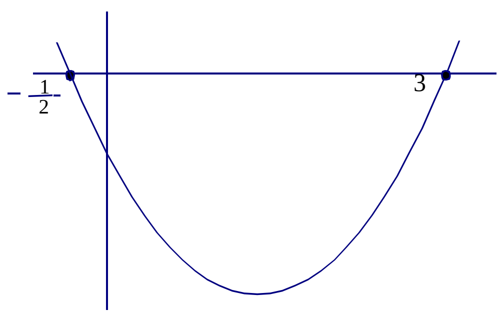
Solving the equation, we get the x-intercepts:

$\Rightarrow 2x + 1 = 0$ or $x - 3 = 0$

$\Rightarrow 2x = -1$ or $x = 3$

$\Rightarrow x = -\frac{1}{2}$ or $x = 3$

Since the graph "opens upward" and the graph crosses the x-axis, the vertex must lie below the x-axis. (See below)



Since the graph of $f(x) = 2x^2 - 5x - 3$ lies below the x-axis between $x = -\frac{1}{2}$ and $x = 3$, this means that $2x^2 - 5x - 3 < 0$ between $x = -\frac{1}{2}$ and $x = 3$. (i.e., $-\frac{1}{2} < x < 3$)

Since the inequality $2x^2 - 5x - 3 < 0$ is equivalent to the original inequality $2x^2 < 5x + 3$, it follows that $2x^2 < 5x + 3$ between $x = -\frac{1}{2}$ and $x = 3$. (i.e., $-\frac{1}{2} < x < 3$)

$-\frac{1}{2} < x < 3$ or $\{x \mid -\frac{1}{2} < x < 3\}$

Makeup 1 Do this exercise only if you lost credit on Exercise 1 on Test #1

Solve the equation: $\frac{x}{x+7} = \frac{9}{8}$

$\Rightarrow 8\left(x+7\right)\frac{x}{x+7} = 8\left(x+7\right)\frac{9}{8}$

$\Rightarrow 8x = \left(x+7\right)\left(9\right)$

$\Rightarrow 8x = 9x + 63$

$\Rightarrow -x = 63$

$\Rightarrow x = -63$

Check: $\frac{(-63)}{(-63)+7} = \frac{-63}{-56} = \frac{(-7)(9)}{(-7)(8)} = \frac{9}{8}$

$\boxed{x = -63}$

Makeup 3 Do this exercise only if you lost credit on Exercise 3 on Test #1

Solve by factoring: $x^2 + 5x - 24 = 0$

$\Rightarrow \left(x+8\right)\left(x-3\right) = 0$

$\Rightarrow x+8 = 0$ or $x-3 = 0$

$\Rightarrow x = -8$ or $x = 3$

Check: $\left(-8\right)^2 + 5\left(-8\right) - 24 = 64 - 40 - 24 = 0$

Check: $\left(3\right)^2 + 5\left(3\right) - 24 = 9 + 15 - 24 = 0$

$\boxed{x = -8 \text{ or } x = 3}$