

MTH 4436 Homework Set 4.3; p. 73 2-7

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2. Prove the following statements:

- (a) For any integer a , the units digit of a^2 is 0, 1, 4, 5, 6, or 9.

The key observation here, is that the units digit of a^2 is congruent to $a^2 \pmod{10}$.

Also, if m is the units digit of n , then \exists a natural number k , such that $n = 10k + m$.
For example, $n = 1295$ can be written as

$$n = (10)(129) + 5$$

At any rate, the units digit of a^2 is equal to $a^2 \pmod{10} \equiv (10k + m)^2 \pmod{10} \equiv m^2 \pmod{10}$.

There are 10 cases:

$m =$	units digit = $m^2 \pmod{10}$
0	0
1	1
2	4
3	9
4	6
5	5
6	6
7	9
8	4
9	1

Thus, the units digit of a^2 is either 0, 1, 4, 5, 6, or 9.

(b) Any one of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 can occur as the units digit of a^3 .

The key observation here, is that the units digit of a^3 is congruent to $a^3 \pmod{10}$.

Also, if m is the units digit of n , then \exists a natural number k , such that $n = 10k + m$.

The units digit of a^3 is equal to $a^3 \pmod{10} \equiv (10k + m)^3 \pmod{10} \equiv m^3 \pmod{10}$.

There are 10 cases:

$m =$	units digit = $m^3 \pmod{10}$
0	0
1	1
2	8
3	7
4	4
5	5
6	6
7	3
8	2
9	9

Thus, the units digit of a^3 can be any one of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.

(c) For any integer a , the units digit of a^4 is 0, 1, 5, or 6.

Again, the key observation here, is that the units digit of a^4 is congruent to $a^4 \pmod{10}$.

Also, if m is the units digit of n , then \exists a natural number k , such that $n = 10k + m$.

The units digit of a^4 is equal to $a^4 \pmod{10} \equiv (10k + m)^4 \pmod{10} \equiv m^4 \pmod{10}$.

There are 10 cases:

$m =$	units digit = $m^4 \pmod{10}$
0	0
1	1
2	6
3	1
4	6
5	5
6	6
7	1
8	6
9	1

Thus, the units digit of a^4 is either 0, 1, 5, or 6.

(d) The units digit of a triangular number is either 0, 1, 3, 5, 6, or 8.

Let t_n be the n^{th} triangular number. As before, the units digit of t_n is congruent to $t_n \pmod{10}$.

By problem 1a, in set 1.1, $t_n = \frac{n(n+1)}{2}$. i.e. $t_n = \frac{n^2+n}{2}$

m	$m^2 \pmod{10}$	$(m^2 + m) \pmod{10}$	$\frac{(m^2+m)}{2} \pmod{10}$
0	0	0	0, 5
1	1	2	1, 6
2	4	6	3, 8
3	9	2	1, 6
4	6	0	0, 5
5	5	0	0, 5
6	6	2	1, 6
7	9	6	3, 8
8	4	2	1, 6
9	1	0	0, 5

Thus, the units digit of a triangular number is either 0, 1, 3, 5, 6, or 8.

3. Find the last two digits of the number 9^{9^9} . [Hint: $9^9 \equiv 9 \pmod{10}$; hence, $9^{9^9} = 9^{9+10k}$; now use the fact that $9^9 \equiv 89 \pmod{100}$]

Observe: $9^{9^9} = 9^{9+10k} = 9^9 \cdot 9^{10k} \equiv 9^9 \cdot (-1)^{10k} \pmod{100} \equiv 9^9 \cdot ((-1)^{10})^k \pmod{100} \equiv 9^9 \cdot 1 \pmod{100} \equiv 9^9 \pmod{100} \equiv 89 \pmod{100}$

Hence, the last two digits of 9^{9^9} are 89.

4. Without performing the divisions, determine whether the integers 176, 521, 221 and 149, 235, 678 are divisible by 9 or 11.

$\boxed{\boxed{176, 521, 221}}$

Observe: $1 + 7 + 6 + 5 + 2 + 1 + 2 + 2 + 1 = 27$, which is divisible by 9.

Since $9 \mid (1 + 7 + 6 + 5 + 2 + 1 + 2 + 2 + 1)$, $9 \mid 176, 521, 221$ also.

Note also, that $1 - 2 + 2 - 1 + 2 - 5 + 6 - 7 + 1 = -3$, which is NOT divisible by 11.

Since $11 \nmid (1 - 2 + 2 - 1 + 2 - 5 + 6 - 7 + 1)$, it follows that $11 \nmid 176, 521, 221$.

$\boxed{\boxed{149, 235, 678}}$

Observe: $1 + 4 + 9 + 2 + 3 + 5 + 6 + 7 + 8 = 45$, which is divisible by 9.

Since $9 \mid (1 + 4 + 9 + 2 + 3 + 5 + 6 + 7 + 8)$, $9 \mid 149, 235, 678$ also.

Note also, that $8 - 7 + 6 - 5 + 3 - 2 + 9 - 4 + 1 = 9$ which is NOT divisible by 11.

Since $11 \nmid (8 - 7 + 6 - 5 + 3 - 2 + 9 - 4 + 1)$, it follows that $11 \nmid 176, 521, 221$.

5. ~

- (a) Obtain the following generalization of Theorem 4.5: If the integer \mathbf{N} is represented in base b by

$$\mathbf{N} = a_m b^m + a_{m-1} b^{m-1} + \dots + a_2 b^2 + a_1 b^1 + a_0 \quad 0 \leq a_k \leq b - 1$$

Then $(b - 1) | \mathbf{N}$ if and only if $(b - 1) | S$, where $S = (a_m + a_{m-1} + \dots + a_2 + a_1 + a_0)$.

Proof. Consider the polynomial with integer coefficients, $P(x) = \sum_{k=0}^m a_k x^k$.

Observe: $b \equiv 1 \pmod{(b - 1)}$.

Hence, $P(b) \equiv P(1) \pmod{(b - 1)}$.

But here's the catch:

$$\begin{aligned} P(b) &= a_m b^m + a_{m-1} b^{m-1} + \dots + a_2 b^2 + a_1 b^1 + a_0 b^0 \\ &= N \end{aligned}$$

and

$$\begin{aligned} P(1) &= a_m (1)^m + a_{m-1} (1)^{m-1} + \dots + a_2 (1)^2 + a_1 (1)^1 + a_0 (1)^0 \\ &= a_m + a_{m-1} + \dots + a_2 + a_1 + a_0 \\ &= S \end{aligned}$$

Now suppose that $(b - 1) | N$.

Since, $N = P(b)$, this is true if and only if $(b - 1) | P(b)$.

$\Leftrightarrow P(b) \equiv 0 \pmod{(b - 1)}$.

Since, $P(b) \equiv P(1) \pmod{(b - 1)}$, this is true if and only if $P(1) \equiv 0 \pmod{(b - 1)}$,

Which, in turn, is true if and only if $(b - 1) | P(1)$,

and this is true if and only if $9 | S$. ■

- (b) Give criteria for the divisibility of N by 3 and 8 that depend on the digits of N when written in base 9.

$\boxed{\boxed{3}}$

Given that $\mathbf{N} = a_m 9^m + a_{m-1} 9^{m-1} + \dots + a_2 9^2 + a_1 9^1 + a_0$,

observe that $(a_m 9^m + a_{m-1} 9^{m-1} + \dots + a_2 9^2 + a_1 9^1) \equiv 0 \pmod{3}$.

Hence, $\mathbf{N} = (a_m 9^m + a_{m-1} 9^{m-1} + \dots + a_2 9^2 + a_1 9^1 + a_0) \equiv a_0 \pmod{3}$.

i.e., $\mathbf{N} \equiv a_0 \pmod{3}$.

Observe:

$3|N \Leftrightarrow \mathbf{N} \equiv 0 \pmod{3} \Leftrightarrow a_0 \equiv 0 \pmod{3} \Leftrightarrow 3|a_0$.

Hence, $3|N$ if and only if $3|a_0$.

$\boxed{\boxed{8}}$

If we let $b = 9$, then $(b - 1) = 8$.

By part a, If the integer \mathbf{N} is represented in base 9 by

$$\mathbf{N} = a_m 9^m + a_{m-1} 9^{m-1} + \dots + a_2 9^2 + a_1 9^1 + a_0 \quad 0 \leq a_k \leq 8$$

Then $8|\mathbf{N}$ if and only if $8|(a_m + a_{m-1} + \dots + a_2 + a_1 + a_0)$.

- (c) Is the number $(447836)_9$ divisible by 3 and 8?

$\boxed{\boxed{3}}$

$a_0 = 6$.

Since $3|a_0$, it follows that $3|(447836)_9$

$\boxed{\boxed{8}}$

Observe: $4 + 4 + 7 + 8 + 3 + 6 = 32_{10}$

Since $8|32_{10}$ it follows that $8|(447836)_9$

6. Working modulo 9 or 11, find the missing digits in the calculations below:

(a) $51840 \cdot 273581 = 1418243x040$.

Observe: $9 \mid (5 + 1 + 8 + 4 + 0)$, hence, $9 \mid (51840)$.

Since $51840 \cdot 273581 = 1418243x040$, it follows that $9 \mid (1418243x040)$.

Hence, $9 \mid (1 + 4 + 1 + 8 + 2 + 4 + 3 + x + 0 + 4 + 0) \Rightarrow 9 \mid (27 + x) \Rightarrow 9 \mid x$

\Rightarrow either $x = 0$ or $x = 9$

Unfortunately, in this case, the “divisibility by 9 criterion” turns out to be inconclusive.

OK, so let’s try using the “divisibility by 11 criterion.”

Observe: $11 \mid (1 - 8 + 5 - 3 + 7 - 2)$, hence $11 \mid 273581$.

Since $51840 \cdot 273581 = 1418243x040$ and 11 divides a factor of the Left Hand side of the equation, it follows that 11 divides the Right Hand side of the equation.

i.e., $11 \mid (1418243x040)$.

Hence, $11 \mid (0 - 4 + 0 - x + 3 - 4 + 2 - 8 + 1 - 4 + 1) \Rightarrow 11 \mid (-13 - x) \Rightarrow 11 \mid (13 + x)$

$\Rightarrow x = 9$

(b) $2x99561 = [3(523 + x)]^2$.

Observe: $2x99561 = [3(523 + x)]^2 = 9(523 + x)^2$

The point of this observation is that $9 \mid 2x99561$.

Hence, $9 \mid (2 + x + 9 + 9 + 5 + 6 + 1) \Rightarrow 9 \mid (32 + x) \Rightarrow x = 4$.

Let’s check: $[3(523 + 4)]^2 = 1581^2 = 2499561$.

Check!

(c) $2784x = x \cdot 5569$

At first glance, it appears that we won't be able to "find the missing digit, working modulo 9 or 11."

If we try applying the "divisibility by 9 criterion," we see that, in order for $2784x$ to be divisible by 9, x must be equal to 6. However, since $(5 + 5 + 6 + 9) = 25$, no value of x , other than $x = 9$, will make $x \cdot 5569$ divisible by 9.

If we try applying the "divisibility by 11 criterion" to the right hand side of the equation:

$$x \cdot 5569$$

we see that, $(9 - 6 + 5 - 5) = 3$.

Hence, 5569 does not have a factor of 11. Therefore, the only value of x that will make $x \cdot 5569$ divisible by 11 is $x = 11$.

Since, from the context of this exercise (see the left hand side of the equation), x is a digit, the conclusion that $x = 11$ is an impossibility.

Thus, the "divisibility by 11 criterion" does not seem to be applicable either.

(d) in order for $2784x$ to be divisible by 11, $11(x - 4 + 8 - 7 + 2) \Rightarrow x = 1$.

However, since $(9 - 6 + 5 - 5) = 3$, it follows that 5569 does not have a factor of 11. Therefore, the only value of x that will make $x \cdot 5569$ divisible by 11 is $x = 11$. no value of x , other than $x = 9$, will make $x \cdot 5569$ divisible by 9.

Let's concentrate on the units digit.

$$2784x = x \cdot 5569 \Rightarrow x \cdot 9 = *x$$

A consideration of the non-zero possibilities shows that $x = 5$:

$$\begin{aligned} 1 \cdot 9 &= 9 \\ 2 \cdot 9 &= 18 \\ 3 \cdot 9 &= 27 \\ 4 \cdot 9 &= 36 \\ 5 \cdot 9 &= 45 \\ 6 \cdot 9 &= 54 \\ 7 \cdot 9 &= 63 \\ 8 \cdot 9 &= 72 \\ 9 \cdot 9 &= 81 \end{aligned}$$

Check: $5 \cdot 5569 = 27845$

Check!

(e) $512 \cdot 1x53125 = 1,000,000,000$

(For future reference, note that $512 = 2^9$. So the only prime factor of 512 is 2.)

Initially, it appears that neither the “divisibility by 9 criterion” nor the “divisibility by 11 criterion” apply here, as

$$9 \nmid (1 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0) \text{ and}$$

$$11 \nmid (0 - 0 + 0 - 0 + 0 - 0 + 0 - 0 + 0 - 1).$$

Perhaps we can manipulate the equation algebraically, so that either the “divisibility by 9 criterion” or the “divisibility by 11 criterion” DOES apply.

Since $9 \nmid (1,000,000,000)$ and $11 \nmid (1,000,000,000)$, we might try to remedy this situation algebraically by adding/subtracting a constant to/from each side of the equation so that the right hand side will be divisible by either 9 or 11.

Our first attempt might be to subtract 1 from the right-hand side of the equation, which will make the right-hand side equal to $999,999,999$ – a number that is clearly divisible by 9.

But when we attempt to subtract 1 from the left-hand side of the equation, we have a problem:

$512 \cdot 1x53125 - 1$ cannot be expressed as a product, so we can't apply the divisibility criteria for 9 and 11 to the left-hand side of the equation.

The lesson that we learn from this is that we must augment/decrement the right-hand side of the equation in increments of 512.

For example:

$$512 \cdot 1x53125 + 512 = 1,000,000,000 + 512$$

$$\Leftrightarrow 512 \cdot (1x53125 + 1) = 1,000,000,512$$

$$\Leftrightarrow 512 \cdot 1x53126 = 1,000,000,512$$

i.e., each increment/decrement of the factor $1x53125$ by 1 on the left-hand side translates into a corresponding increment/decrement of the right-hand side by $1,000,000,000$ by 512.

Sooooo . . . incrementing both sides in this fashion yields:

$$512 \cdot 1x53126 = 1,000,000,512$$

The right-hand side is clearly divisible by 9, as $9 \mid (1 + 0 + 0 + 0 + 0 + 0 + 0 + 5 + 1 + 2)$

Hence, $9 \mid (512 \cdot 1x53126) \Rightarrow 9 \mid 1x53126$, since $9 \nmid 512$.

$$\Rightarrow 9 \mid (1 + x + 5 + 3 + 1 + 2 + 6) \Rightarrow 9 \mid (x + 18) \Rightarrow x = 0 \text{ or } x = 9.$$

The divisibility by 9 criterion proves to be inconclusive.



Soooo . . . let's increment both sides again. This will yield:

$$512 \cdot 1x53127 = 1,000,001,024$$

The right-hand side is clearly divisible by 11, as $11 \mid (4 - 2 + 0 - 1 + 0 - 0 + 0 - 0 + 0 - 1)$

Hence, $11 \mid (512 \cdot 1x53127) \Rightarrow 11 \mid 1x53127$, since $11 \nmid 512$.

$$\Rightarrow 11 \mid (7 - 2 + 1 - 3 + 5 - x + 1) \Rightarrow 11 \mid (9 - x) \Rightarrow x = 9$$

$$\text{Check: } 512 \cdot 1953125 = 2^9 \cdot 5^9 = (2 \cdot 5)^9 = 1,000,000,000$$

Check!

$$x = 9.$$

7. Establish the following divisibility criteria:

- (a) An integer is divisible by 2 if and only if its units digit is 0, 2, 4, 6, or 8.

Proof. Let N be the integer under consideration, and let x be the units digit of N .

Note that N can be expressed as

$$N = k \cdot 10 + x$$

where $k \in \mathbf{Z}$, and $x = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

Note that $k \cdot 10 \equiv 0 \pmod{2}$.

Hence, $N \equiv x \pmod{2}$.

Observe:

$$2|N$$

$$\Leftrightarrow N \equiv 0 \pmod{2}$$

$$\Leftrightarrow x \equiv 0 \pmod{2}$$

$$\Leftrightarrow x \in \{0, 2, 4, 6, 8\}. \blacksquare$$

- (b) An integer is divisible by 3 if and only if the sum of its digits is divisible by 3.

(i.e., if $N = a_m a_{m-1} \dots a_2 a_1 a_0$, then $3|N \Leftrightarrow 3|(a_m + a_{m-1} + \dots + a_2 + a_1 + a_0)$.)

Proof. Define $P(x) = a_m x^m + a_{m-1} x^{m-1} + \dots + a_2 x^2 + a_1 x + a_0$.

Note: Since $10 \equiv 1 \pmod{3}$, it follows that $P(10) \equiv P(1) \pmod{3}$.

Observe:

$$3|N \Leftrightarrow N \equiv 0 \pmod{3} \Leftrightarrow P(10) \equiv 0 \pmod{3} \Leftrightarrow P(1) \equiv 0 \pmod{3} \Leftrightarrow 3|P(1) \Leftrightarrow 3|(a_m + a_{m-1} + \dots + a_2 + a_1 + a_0). \blacksquare$$

- (c) An integer is divisible by 4 if and only if the number formed by its tens and units digits is divisible by 4.

Proof. Let $N = a_m a_{m-1} \dots a_2 a_1 a_0$. Note that N can be written as:

$$N = k \cdot 10^2 + a_1 a_0.$$

Since $k \cdot 10^2 \equiv 0 \pmod{4}$, it follows that $N = k \cdot 10^2 + a_1 a_0 \equiv a_1 a_0 \pmod{4}$.

i.e., $N \equiv a_1 a_0 \pmod{4}$.

Observe:

$$4|N$$

$$\Leftrightarrow N \equiv 0 \pmod{4}$$

$$\Leftrightarrow a_1 a_0 \equiv 0 \pmod{4}$$

$$\Leftrightarrow 4|a_1 a_0. \blacksquare$$

- (d) An integer is divisible by 5 if and only if its units digit is 0 or 5.

Proof. Let N be the integer under consideration, and let x be the units digit of N .

Note that N can be expressed as

$$N = k \cdot 10 + x$$

where $k \in \mathbf{Z}$, and $x = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

Note that $k \cdot 10 \equiv 0 \pmod{5}$.

Hence, $N \equiv x \pmod{5}$.

Observe:

$$5|N$$

$$\Leftrightarrow N \equiv 0 \pmod{5}$$

$$\Leftrightarrow x \equiv 0 \pmod{5}$$

$$\Leftrightarrow x \in \{0, 5\}. \blacksquare$$