MTH 4436 Homework Set 3.2; page 49

Spring 2015

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1. Determine whether the integer 701 is prime by testing all primes $p \leq \sqrt{701} = 26.5$ as possible divisors. Do the same for the integer 1009.

$701 = 2(350) + 1$

$701 = 3(233) + 2$

$701 = 5(140) + 1$

$701 = 7(100) + 1$

$701 = 11(63) + 8$

$701 = 13(53) = 12$

$701 = 17(41) + 4$

$701 = 19(36) + 17$

$701 = 23(30) + 11$

No prime numbers less than $\sqrt{701}$ are divisors of 701. Therefore, 701 is prime.

For 1009, consider all primes $p \leq \sqrt{1009} = 31.765$

$1009 = 2(504) + 1$

$1009 = 3(336) + 1$

$1009 = 5(201) + 4$

$1009 = 7(144) + 1$

$1009 = 11(91) + 8$

$1009 = 13(77) + 8$

$1009 = 17(59) + 6$

$1009 = 19(53) + 2$

$1009 = 23(43) + 20$

$1009 = 29(34) + 23$

$1009 = 31(32) + 17$

No prime numbers less than $\sqrt{1009}$ are divisors of 1009. Therefore, 1009 is prime.

2. Employing the sieve of Eratosthenes, obtain all the primes between 100 and 200.

	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100
101	102	103	104	105	106	107	108	109	110
111	112	113	114	115	116	117	118	119	120
121	122	123	124	125	126	127	128	129	130
131	132	133	134	135	136	137	138	139	140
141	142	143	144	145	146	147	148	149	150
151	152	153	154	155	156	157	158	159	160
161	162	163	164	165	166	167	168	169	170
171	172	173	174	175	176	177	178	179	180
181	182	183	184	185	186	187	188	189	190
191	192	193	194	195	196	197	198	199	200

Weeding out the multiples of the primes, we have:

	2	3		5		7			
11		13				17		19	
		23						29	
31						37			
41		43				47			
		53						59	
61						67			
71		73						79	
		83						89	
						97			
101		103				107		109	
		113							
						127			
131						137		139	
								149	
151						157			
		163				167			
		173						179	
181									
191		193				197		199	

Thus, we find the primes between 100 and 200, to be: $101, 103, 107, 109, 113, 127, 131,$

$137, 139, 149, 151, 157, 163, 167, 173, 179, 181, 191, 193, 197, 199.$

3. Given that $p \nmid n$ for all primes $p \leq \sqrt[3]{n}$, show that $n > 1$ is either a prime, or the product of exactly two primes.

   Proof. Let the hypothesis be given. If n is prime, then our proposition is proved, so assume that n is composite. Suppose, for the sake of contradiction, that n is the product of at least three prime factors. (i.e., $n = p_1 p_2 \ldots p_k$ with $k \geq 3$.)

   By hypothesis, $p_i > \sqrt[3]{n}$ for $i = 1, 2, 3, \ldots, k$.

   Observe: $n = p_1 p_2 \ldots p_k > \underbrace{\sqrt[3]{n}\sqrt[3]{n}\ldots\sqrt[3]{n}}_{k \text{ factors}} = \left(\sqrt[3]{n}\right)^3 \left(\sqrt[3]{n}\right)^{k-3} = n \cdot \left(\sqrt[3]{n}\right)^{k-3} \geq n.$

   i.e., $n > n$, a contradiction.

   Hence, n is the product of at most two primes. ■

4. ~

(a) Show that $\sqrt{p}$ is irrational for all primes p.

Proof. (By Contradiction) Suppose that p is prime.

For the sake of deriving a contradiction, suppose that $\sqrt{p}$ is rational.

Then $\exists\ m, n \in \mathbb{N}$ such that $\sqrt{p} = \frac{m}{n}$.

Without loss of generality, we can assume that m and n are relatively prime (i.e., $\frac{m}{n}$ is "reduced to lowest terms")

Observe: $\sqrt{p} = \frac{m}{n}$

$\Rightarrow p = \frac{m^2}{n^2}$

$\Rightarrow pn^2 = m^2 \quad$ (eq.1)

$\Rightarrow p|m^2$

$\Rightarrow p|m$ (by Thm 3.1)

$m = kp$ for some $k \in \mathbb{N}$

Thus, eq. 1 becomes:

$pn^2 = (kp)^2$

$\Rightarrow pn^2 = k^2p^2$

$\Rightarrow n^2 = k^2p$

$\Rightarrow p|n^2$

$\Rightarrow p|n$ (by Thm 3.1)

But this contradicts the premise that m and n are relatively prime.

Since the assumption that $\sqrt{p}$ is rational leads to a contradiction, the assumption is false.

Hence, $\sqrt{p}$ is irrational. ■

(b) If a is a positive integer and $\sqrt[n]{a}$ is rational, then $\sqrt[n]{a}$ must be an integer.

Proof. Let the hypotheses be given.

(i.e., Suppose that a is a positive integer and $\sqrt[n]{a}$ is rational.)

Since $\sqrt[n]{1} = 1$, the proposition is true for $a = 1$.

So, suppose that $a \geq 2$.

Also, for the sake of deriving a contradiction, suppose that $\sqrt[n]{a}$ is not an integer.

Then $\exists\, k, m \in \mathbb{N}$, with $m \geq 2$ such that $\sqrt[n]{a} = \frac{k}{m}$.

Without loss of generality, we can assume that k and m are relatively prime (i.e., $\frac{k}{m}$ is "reduced to lowest terms")

Observe: $\sqrt[n]{a} = \frac{k}{m}$

$\Rightarrow a = \frac{k^n}{m^n}$

$\Rightarrow m^n a = k^n$

$\Rightarrow m^n | k^n$

$\Rightarrow m | k^n \quad$ (eq.1)

By the Fundamental Thm of Arithmetic, m has at least one prime factor p.

$\Rightarrow m = jp$ for some $j \in \mathbb{N}$.

Plugging this into eq.1, we have
$(jp)\,|k^n$

$\Rightarrow p|k^n$

$\Rightarrow p|k$ (by Thm 3.1)

However, this contradicts the assumption that k and m are relatively prime.

Since the assumption that $\sqrt[n]{a}$ is not an integer leads to a contradiction, the assumption must be false.

Hence, $\sqrt[n]{a}$ is an integer. ■

(c) For $n \geq 2$, $\sqrt[n]{n}$ is irrational.

Proof. Let the hypotheses be given.

(i.e., Suppose that $n \in \mathbb{N}$ and $n \geq 2$.)

First, we make an observation: $\forall\ n \in \mathbb{N};\ n < 2^n$

This can easily be proved by induction.

First note that the proposition is true for $n = 1$, as $1 < 2 = 2^1$.

Next, suppose that for some $n = k$, $k < 2^k$

Observe that $\underbrace{k + 1 < 2^k + 2^k}_{\text{by Ind. Hyp}} = 2 \cdot 2^k = 2^{k+1}$

i.e. $(k + 1) < 2^{k+1}$ (End of Observation)

Based on our observation, $\forall n \geq 2;\ 1 < \sqrt[n]{n} < \sqrt[n]{2^n} = 2$.

(i.e., $1 < \sqrt[n]{n} < 2$.)

By the *contrapositive* of the result in part b, if a is a positive integer and $\sqrt[n]{a}$ is *not* an integer, then $\sqrt[n]{a}$ is irrational.

Since $1 < \sqrt[n]{n} < 2$, it follows that $\sqrt[n]{n}$ cannot be an integer.

Hence, $\sqrt[n]{a}$ must be irrational. ■

5. Show that any composite, three digit number must have a prime factor less than or equal to 31.

Proof. Let n be a composite, three digit number. Since $31^2 = 961$ and $32^2 = 1024$, it must be the case that $\sqrt{n} < \sqrt{1000} < 32$.

(i.e., $\sqrt{n} \leq 31$.)

Since every composite number must have a prime factor less than or equal to its own square root, n must have a prime factor less than or equal to 31.

(To see a proof of this last statement, let $n = ab$, where $a \leq b$.

Then $a^2 \leq ab = n$.

i.e., $a^2 \leq n \Rightarrow a \leq \sqrt{n}$.) ■