Homework - Sec. 2.3, Page 25

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1. If $a|b$, show that $(-a)\,|b$, $a|\,(-b)$, and $(-a)\,|\,(-b)$.

   Let the hypothesis be given (i.e., suppose that $a|b$).

   Then $\exists$ an integer m such that $b = am$.

   $\Rightarrow b = (-a)\,(-m) \Rightarrow (-a)\,|b$.

   Also, the fact that $b = am$ implies that $(-b) = a\,(-m)$, hence, $a|\,(-b)$.

   Finally, the fact that $b = am$ implies that $(-b) = (-a)\,(m)$, hence, $(-a)\,|\,(-b)$. $\blacksquare$

2. Given integers a, b, c, d, verify the following:

   (a) If $a|b.$ then $a|bc.$

   Let the hypothesis be given (i.e., suppose that $a|b$).
   Then $\exists$ an integer m such that $b = am$.
   $\Rightarrow bc = (am)\,c = a\,(mc)$
   i.e., $bc = a\,(mc)$.
   Hence, $a|bc$. $\blacksquare$

   (b) If $a|b$ and $a|c,$ then $a^2|bc.$

   Let the hypothesis be given (i.e., suppose that $a|b$ and $a|c$).
   Then $\exists$ an integer(s) m and n such that $b = am$ and $c = an$.
   Then $bc = (am)\,(an) = a^2\,(mn)$.
   i.e., $bc = a^2\,(mn)$, and hence, $a^2|bc$. $\blacksquare$

   (c) $a|b$ if and only if $ac|bc,$ where $c \neq 0.$

   $\boxed{a|b \Rightarrow ac|bc}$
   Suppose that $a|b$. Then $\exists$ an integer m such that $b = am$.
   Observe: $bc = (am)\,c = (ac)\,m$
   i.e., $bc = (ac)\,m,$ and hence, $ac|bc$.
   $\boxed{ac|bc \Rightarrow a|b}$
   Suppose that $ac|bc$. Then $\exists$ an integer m such that $bc = (ac)\,m$.
   $\Rightarrow bc = (am)\,c$.
   Since $c \neq 0$, we can divide both sides by zero.
   $\Rightarrow b = am$.
   Hence, $a|b$. $\blacksquare$

(d) If $a|b$ and $c|d$, then $ac|bd$

Let the hypothesis be given (i.e., suppose that $a|b$ and $c|d$).
Then $\exists$ an integer(s) m and n such that $b = am$ and $d = cn$.
Then $bd = (am)(cn) = (ac)(mn)$.
i.e., $bd = (ac)(mn)$, and hence, $ac|bd$. ■

3. Prove or disprove: If $a|(b+c)$, then either $a|b$ or $a|c$.

This is false. To show that it's false, we need to offer a counterexample.

Consider: $a = 2$, $b = 3$, and $c = 5$.

$a|(b+c)$, but $a \nmid b$ and $a \nmid c$.

5. Prove that for any integer a, one of the integers, $a, a+2, a+4$ is divisible by 3.

Let a be any integer. Then by the division algorithm, there are exactly three mutually exclusive and mutually exhaustive possibilities — either $a = 3k$, $a = 3k+1$, or $a = 3k+2$.

Case 1: $a = 3k$

If $a = 3k$, then our assertion is proved.

Case 2: $a = 3k+1$

If $a = 3k+1$, then $a+2 = (3k+1)+2 = 3k+3 = 3(k+1)$

i.e., $a+2 = 3(k+1)$. Hence, $3|(a+2)$

Case 2: $a = 3k+2$

If $a = 3k+2$, then $a+4 = (3k+2)+4 = 3k+6 = 3(k+2)$

i.e., $a+4 = 3(k+2)$. Hence, $3|(a+4)$

Since all possibilities have been exhausted and since our proposition is true for each possibility, we can say that for any integer a, one of the integers, $a, a+2, a+4$ is divisible by 3. ■

8. ~

(a) The sum of the squares of two odd integers cannot be a perfect square.

Two arbitrary odd integers can be represented as $2k+1$ and $2j+1$.
The sum of their squares is $(2k+1)^2 + (2j+1)^2 = 4k^2+4k+1+4j^2+4j+1 = 4k^2+4k+4j^2+4j+2 = 4\left(k^2+k+j^2+j\right)+2 = 4m+2$, where $m = k^2+k+j^2+j$
The point here, is that the sum of the squares of any two odd numbers is of the form $4m+2$.
Could this be a perfect square? Well, let's see what form(s) perfect squares can have.
An even number $2n$, when squared, has the form: $4n^2 = 4m$, where $m = n^2$.
An odd number $2n+1$, when squared, has the form: $4n^2+4n+1 = 4m+1$, where $m = n^2+n$.
What we have learned is that perfect squares are either of the form $4m$ or $4m+1$. Perfect squares are *never* of the form $4m+2$, which is the form that the sum of the squares of two odd numbers always has.
Hence, the sum of the squares of two odd integers cannot be a perfect square. ■

(b) The product of four consecutive integers is one less than a perfect square.

We can represent the four consecutive integers as $n, n+1, n+2, n+3$.
Their product is $n(n+1)(n+2)(n+3) = n^4+6n^3+11n^2+6n$
This is one less than $n^4+6n^3+11n^2+6n+1 = \left(n^2+3n+1\right)^2$, which is a perfect square. ■

Alternatively, Alex Leveque offered this proof:

To say that "The product of four consecutive integers is one less than a perfect square" is equivalent to saying that "The product of four consecutive integers can be written in the form a^2-1."

Since a^2-1 can be written as $(a-1)(a+1)$, we are motivated to prove our claim by expressing The product of four consecutive integers in the form $(a-1)(a+1)$.

Observe: We can represent the four consecutive integers as $n, n+1, n+2, n+3$.

Their product is:

$$n(n+1)(n+2)(n+3) = n(n+3)(n+1)(n+2) = \left(n^2+3n\right)\left(n^2+3n+2\right)$$

$$= \underbrace{\left[\underbrace{\left(n^2+3n+1\right)}_{=a} - 1\right]}_{a-1}\underbrace{\left[\underbrace{\left(n^2+3n+1\right)}_{=a} + 1\right]}_{a+1} = \underbrace{\underbrace{\left(n^2+3n+1\right)^2}_{=a^2} - 1}_{a^2-1}$$

i.e., $n(n+1)(n+2)(n+3) = a^2-1$, where $a = n^2+3n+1$ ■

9. Establish that the difference of two consecutive cubes is never divisible by 2.

    Case 1: The smaller number is even.

    Let a be even. Then $\exists$ a natural number k such that $a = 2k$. This being the case, $a + 1 = 2k + 1$

    The difference of the cubes of these numbers is $(a+1)^3 - a^3 = (2k+1)^3 - (2k)^3 = 12k^2 + 6k + 1 = \underbrace{2\left(6k^2 + 3k\right) + 1}_{2m+1}$, which is odd.

    Case 2: The smaller number is odd.

    Let a be odd. Then $\exists$ a natural number k such that $a = 2k + 1$. This being the case, $a + 1 = 2k + 2$

    The difference of the cubes of these numbers is $(a+1)^3 - a^3 = (2k+2)^3 - (2k+1)^3 = 12k^2 + 18k + 7 = \underbrace{2\left(6k^2 + 9k + 3\right) + 1}_{2m+1}$, which is odd.

    Since this exhausts all cases, and each case results in the difference of two consecutive cubes being odd, our assertion is proved. ■

11. If a and b are integers, not both of which are zero, verify that:

$$\gcd(a,b) = \gcd(-a,b) = \gcd(a,-b) = \gcd(-a,-b).$$

$d = \gcd(a,b)$

$\Leftrightarrow d$ is the smallest natural number such that $\exists x, y \in \mathbf{Z}$ such that $ax + by = d$

$\Leftrightarrow d$ is the smallest natural number such that$\exists x_1, y \in \mathbf{Z}$ such that

$$\underbrace{(-a)\,x_1}_{x_1=-x} + by = d \qquad \text{(Eq.1)}$$

(and consequently $d = \gcd(-a,b)$)

$\Leftrightarrow d$ is the smallest natural number such that $\exists x_1, y_1 \in \mathbf{Z}$ such that

$\underbrace{(-a)\,x_1}_{x_1=-x} + \underbrace{(-b)\,y_1}_{y_1=-y} = d$

(and consequently $d = \gcd(-a,-b)$)

$\Leftrightarrow d$ is the smallest natural number such that $\exists x, y_1 \in \mathbf{Z}$ such that $\underbrace{(a)\,x}_{x=-x_1} + \underbrace{(-b)\,y_1}_{y_1=-y} = d.$

(and consequently $d = \gcd(a,-b)$) ■

Remark 1 *In Eq. 1, if d were NOT the smallest natural number such that* $\underbrace{(-a)\,x_1}_{x_1=x} + by = d$*, then there must be a smaller natural number c such that* $\underbrace{(-a)\,x_1}_{x_1=x} + by = c.$

But this would imply that $\underbrace{ax_1}_{x_1=x} + by = c$*, contradicting the assumption that d was the smallest such natural number.*

Remark 2 *This was not the solution that I originally had. Julian Allagan showed me this proof and I liked it, so I changed the one that I had.*

18a. Prove: The product of any three consecutive integers is divisible by 6.

Let the integers be represented as n, $n+1$, $n+2$.

Claim: At least one of the integers must be even.

By the Division Algorithm, n is either of the form $2k$ or $2k+1$.

If n is of the form $2k$, our claim is proved.

If n is of the form $2k+1$, then $n+1 = (2k+1)+1 = 2(k+1)$, and hence, $n+1$ is even.

End of Claim

Claim: At least one of the integers must be divisible by 3.

By the Division Algorithm, n is either of the form $3k$, $3k+1$, or $3k+2$.

If n is of the form $3k$, then our claim is proved.

If n is of the form $3k+1$, then $n+2 = (3k+1)+2 = 3(k+1)$, and hence, $n+2$ is divisible by 3.

If n is of the form $3k+2$, then $n+1 = (3k+2)+1 = 3(k+1)$, and hence, $n+1$ is divisible by 3.

End of Claim

Thus $2|(n)(n+1)(n+2)$ and $3|(n)(n+1)(n+2)$.

Since $\gcd(2,3) = 1$, the second corollary to Theorem 2.4 tells us that

$$(2 \cdot 3) \mid (n)(n+1)(n+2).$$

i.e., the product of any three consecutive integers is divisible by 6. ■

18b. Prove: The product of any four consecutive integers is divisible by 24.

Let the integers be represented as n, $n+1$, $n+2$, and $n+3$.

By our results in part 11a, the product $(n)(n+1)(n+2)(n+3)$ is divisible by 3.

Claim: Our product is divisible by 8.

By the Division Algorithm, n is either of the form $4k$, $4k+1$, $4k+2$, or $4k+3$.

If n is of the form $4k$, then $4k+2 = 2(2k+1)$, and the product $(n)(n+1)(n+2)(n+3) = (4k)(4k+1)(4k+2)(4k+3) = (4k)(4k+1)2(2k+1)(4k+3)$

$= 8k(4k+1)(2k+1)(4k+3)$. Our claim is proved.

If n is of the form $4k+1$, then $n+1 = (4k+1)+1 = 2(2k+1)$. Furthermore, $n+3 = (4k+1)+3 = 4(k+1)$.

Hence, our product $(n)(n+1)(n+2)(n+3) = (4k+1)(4k+2)(4k+3)(4k+4) = (4k+1)2(2k+1)(4k+3)4(k+1) = 8(4k+1)(2k+1)(4k+3)(k+1)$.

And our claim is proved.

Similar arguments can be used to show that if $n = 4k+2$ or $n = 4k+3$, then $8|(n)(n+1)(n+2)(n+3)$

End of Claim

We have established that $8|(n)(n+1)(n+2)(n+3)$ and that $3|(n)(n+1)(n+2)(n+3)$

Since $\gcd(3,8) = 1$, the second corollary to Theorem 2.4 tells us that

$$(3 \cdot 8) \mid (n)(n+1)(n+2)(n+3).$$

i.e., the product of any four consecutive integers is divisible by 24. ■

18c. Prove: The product of any five consecutive integers is divisible by 120.

By our results in part 11b, the product $(n)(n+1)(n+2)(n+3)(n+4)$ is divisible by 24.

Since $\gcd(5,24) = 1$, it remains to show that $5|(n)(n+1)(n+2)(n+3)(n+4)$.

An argument similar to those of part 11a., can be used to show this.

Hence, the product of five consecutive integers is divisible by 120. ■